Complex Analysis

Chao Ren

December 20, 2025

Contents

Chapter 0

Introduction

0.3 Basic notations

Definition 1. We say that $U \subset C$ is *open* if for every $z \in U$ there is some $\varepsilon > 0$ such that an open ball $B(z, \varepsilon)\{w \in \mathbb{C} : |z - w| < \varepsilon\} \subset U$. Any open set U containing z is called a *neghborhood* of z.

Definition 2. A connected open subset $D \subseteq \mathbb{C}$ of the complex plane will be called a *domain*.

Chapter 1

Complex Differentiability

1.1 Complex Differentiability

Definition 3. (Complex Differentiability)

Example. Suppose we have a functor. If $G_X \ncong G_Y$, then X and Y are not homeomorphic. If 'shadows' are different, then objects themselves are different too.

Proof. Suppose X and Y are homeomorphic. Then $\exists f : X \to Y$ and $g : Y \to X$, maps (maps are always continuous in this course), such that $g \circ f = 1_X$ and $f \circ g = 1_Y$. Then $f_* : G_X \to G_Y$ and $g_* : G_Y \to G_X$ such that $(g \circ f)_* = (1_X)_*$ and $(f \circ g)_* = (1_Y)_*$. Using the rules discussed previously, we get

$$g_* \circ f_* = 1_{G_X} \quad f_* \circ g_* = 1_{G_Y},$$

which means that $f_* : G_X \to G_Y$ is an isomorphism.

Chapter 3

Cauchy's Formula and its Applications

3.1 Cauchy's Integral Formula

Theorem 1 (Cauchy's Integral Formula)**.** Suppose that $f : U \to \mathbb{C}$ is a holomorphic function on an open set U, $w \in U$ and γ is a simple positively oriented closed curve such that γ^* and the interior of γ are inside of U. Then for all w that are inside of γ w have

$$f(w) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-w} dz.$$

Remark. Same result holds for any oriented curve γ once we weigh the LHS by the winding number of a path around the point $w \notin \gamma^*$, provided that f is holomorphic on the inside of γ.

Corollary (Cauchy Formula for multiple curves)**.** Let U be a bounded domain with piecewise C^1 boundary which has finitely many components and f be a function holomorphic in the closure of U (this means that it is holomorphic in some open domain that contains the closure of U). We parametrize each boundary component of U by a contour γ_i in such way that $i\gamma_i'(t)$ is an inward normal. This means that the 'outer' boundary is positively oriented (i.e. counter-clockwise) and all 'inner' components are negatively oriented (i.e. clockwise). Denoting $\int_{\delta U} = \sum \int_{\gamma_i}$ we have

$$\int_{\delta U} f(z)dz = 0$$

and

$$\frac{1}{2\pi i} \int_{\delta U} \frac{f(z)}{z-w} dz = f(w), w \in U.$$

3.2 Homotopy Version of Cauchy's Theorem

NON-EXAMINABLE

3.3 Applications of the Integral Formula

Corollary. If $f : U \to \mathbb{C}$ is holomorphic on an open set U, then for any $z_0 \in U$, $f(z)$ is equal to its Taylor series at z_0 and the Taylor series converges on any open disk centered at z_0 lying in U. Moreover the derivatives of f at z_0 are given by

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\gamma(z_0,r)} \frac{f(z)}{(z-z_0)^{n+1}} dz$$

Definition 4. A function which is locally given by a power series is said to be *analytic*. We have thus shown that any holomorphic function is actually analytic, and from now on we may use the terms interchangeably.

Definition 5. A function $f : \mathbb{C} \to \mathbb{C}$ is *entire* if it is complex differentiable on the whole complex plane.

Theorem 2 (Liouville)**.** Let $f : \mathbb{C} \to \mathbb{C}$ be an entire function. If f is bounded then it is constant.

Remark. Liouville's theorem is another manifestation of the unique properties of holomorphic functions. In real analysis, $f(x) = \frac{1}{1+x^2}$ is real-analytic in the entire $\mathbb{R}$, but it is bounded. **This is also one of the first examples of a dichotomy which often appears in complex analysis.** In many cases objects are as good or as bad as they could be but nothing in between.

Theorem 3. Suppose that $p(z) = \sum_{k=0}^{n} a_k z^k$ is a non-constant polynomial where $a_k \in \mathbb{C}$ and $a_n \neq 0$. Then there is a $z_0 \in \mathbb{C}$ for which $p(z_0) = 0$.

Remark. The crucial point of the above proof is that on term of the polynomial dominates the behavior for large z. All proofs of the fundamental theorem hinge on essentially this point.

Theorem 4. Suppose $f : U \to \mathbb{C}$ is a continuous function on an open subset $U \subseteq \mathbb{C}$. If for any closed path $\gamma : [a, b] \to oU$ we have $\int_\gamma f(z)dz = 0$, then f is holomorphic.

3.4 The Identity Theorem

3.5 Isolated Singularities

Definition 6. Let $f : U \to \mathbb{C}$ be a function, where U is open. We say that $z_0 \in \overline{U}$ is a *regular* point of f if f is holomorphic at z_0. Otherwise we say that z_0 is *singular*.
We say that z_0 is an *isolated singularity* if f is holomorphic on $B(z_0, r)\backslash\{z_0\}$ for some $r > 0$.

Definition 7. A function on an open set U which has only isolated singularities all of which are poles is called a *meromorphic* function on U. (Strictly, it is a function only defined on the complement of the poles in U.)

Definition 8. Let z_0 be an isolated singularity of function f. We say that z_0 is

- A *removable singularity* if there is a function g holomorphic in $B(z_0, r)$ for some $r > 0$ such that such that $f(z) = g(z)$ in $B(z_0, r) \setminus \{z_0\}$.
- A *pole of order* n if there is a function g holomorphic in $B(z_0, r)$ for some $r > 0$ such that $g(z_0) \neq 0$ and $f(z) = (z - z_0)^{-n} g(z)$ in $B(z_0, r) \setminus \{z_0\}$.

Example. Let $f(z) = \frac{\sin z}{z}$. $z_0 = 0$ is a removable singularity.

Proof. The function

$$g(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

is entire and coincides with f outside of z_0.

Example. Let $f(z) = \frac{\sin z}{z^{n+1}}$. This function has a pole of order n at z_0.

Proof. It is easy to see for $z \neq 0$ we have $f(z) = \frac{g(z)}{z^n}$ where g is the entire function from the previous example. Obviously $g(0) = 1 \neq 0$.

Example. Let $f(z) = \sin(\frac{1}{z})$. This function is holomorphic in $C \setminus \{0\}$. It can be shown 0 is neither a removable singularity nor a pole, so it must be an essential singularity.

Theorem 5. (Laurent's Theorem) Suppose that $0 < r < R$ and

$$A = A(z_0, r, R) = \{z : r < |z - z_0| < R\}$$

is an annulus centered at z_0. If $f : U \to \mathbb{C}$ is holomorphic on an open set U which contains $\overline{A}$, then there exist $c_n \in \mathbb{C}$ such that

$$\sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$$

Th series converges for all $z \in A$ and it converges uniformly for all $z \in A(z_0, r', R')$ where $r < r' < R' < R$. The series is called the *Laurent series* of f. Moreover, the c_n are unique and are given by the following formulae:

$$c_n = \frac{1}{2\pi i}\int_{\gamma_s} \frac{f(z)}{(z-z_0)^{n+1}}dz,$$

where $s \in [r, R]$ and for any $s > 0$ we set $\gamma_s(t) = z_0 + se^{2\pi it}$.

Remark. Given a formula for c_n in terms of f, the Laurent expansion is unique. If two series converge to the same function then they coincide term-by-term.

Remark. If f is holomorphic in $B(z_0, R)$, then for $n < 0$ the integrand in the formula for c_n is holomorphic, hence $c_n = 0$ for all $n < 0$. For $n \geq 0$ formulas for c_n are exactly the same as in Taylor's theorem so in this case the Laurent series is the same as the Taylor series.

Corollary. If $f : U \to \mathbb{C}$ is a holomorphic function and z_0 is an isolated singularity, then f has a Laurent expansion on punctured disc $B(z_0, R) \setminus \{z_0\}$ for any R such that $B(z_0, R) \setminus \{z_0\} \subset U$.

Definition 9. Let z_0 be an isolated singularity of f and $\sum c_n(z-z_0)^n$ be its Laurent expansion. Its *principal part* of f at z_0 is the sum of terms with negative powers and denoated $P_{z_0}f$. Namely,

$$P_{z_0}f = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n = \sum_{n=1}^{\infty} c_{-n}(z-z_0)^n$$

Th principal part of f at z_0 converges on $\mathbb{C} \setminus \{z_0\}$ and converges uniformly on $\mathbb{C} \setminus B(z_0, r)$.

Definition 10. Let z_0 be an isolated singularity of f. Then the *residue* of f at z_0 is defined as the coefficient c_{-1} of the Laurent expansion and denoted by $Res_{z_0}f$ or $Res(f, z_0)$.

Theorem 6. Let z_0 be an isolated singularity of f. Let $\sum_{-\infty}^{\infty} c_n(z-z_0)^n$ be its Laurent expansion. Then z_0 is

- A removable singularity if $c_n = 0$ for all $n < 0$. Equivalently, the principal part vanishes.
- A pole of order n if $c_{-n} \neq 0$ and $c_k = 0$ for all $k < -n$. Equivalently, the principal part is non-trivial but contains only a finite number of non-zero terms.
- An essential singularity if there are arbitrarily large n such that $c_{-n} \neq 0$. Equivalently, the principal part contains finitely main non-zero terms.

Theorem 7. (Riemann's removable singularity theorem) Suppose that U is an open subset of $\mathbb{C}$ and $z_0 \in U$ and suppose that $f : U \setminus \{z_0\} \to \mathbb{C}$ is holomorphic. Then z_0 is a removable singularity if and only if f is bounded near z_0.

Lemma 1. Let f be a holomorphic function in a neighborhood of z_0. Then z_0 is a pole if and only if $|f(z)| \to \infty$ as $z \to z_0$. Moreover, in this case, the function

$$h(z) = \begin{cases} \frac{1}{f(z)}, & z \neq z_0; \\ 0, & z = z_0; \end{cases}.$$

is holomorphic in a neighborhood of z_0 and the multiplicity of its zero at z_0 is equal to the order of the pole of f.

Theorem 8. (Casorati-Weierstrass) Let U be an open subset of $\mathbb{C}$ and let $a \in U$. Suppose that $f : U \setminus \{a\} \to \mathbb{C}$ is a holomorphic function with an isolated essential singularity at a. Then for all $\rho > 0$ with $B(a, \rho) \subseteq U$, the set $f(B(a, a\rho) \setminus \{a\}$ is dense in $\mathbb{C}$, that is, the closure of $f(B(a, \rho) \setminus \{a\})$ is all of $\mathbb{C}$.

Theorem 9. (Residue Theorem) Suppose that U is an open set in $\mathbb{C}$ and γ is a closed curve that is contained in U together with its inside. Suppose that f is holomorphic on $U \setminus S$ where S is a finite set of isolated singularities of f. We also assume that f has no singularities on $\gamma\star$, that is $S \cap \gamma\star = \emptyset$. Then

$$\frac{1}{2\pi i} \int_\gamma f(z)dz = \sum_{a \in S} I(\gamma, a) Res_a(f).$$